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Generalized Gaudin systems in a magnetic field and non-skew-symmetric *r*-matrices

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Abstract
We construct integrable cases of generalized classical and quantum Gaudin spin chains in an external magnetic field. For this purpose, we generalize the 'shift of argument method' onto the case of classical and quantum integrable systems governed by an arbitrary $\mathfrak{g}$-valued, non-dynamical classical r-matrix with spectral parameters. We consider several examples of the obtained construction for the cases of skew-symmetric, 'twisted' non-skew-symmetric and 'anisotropic' non-skew-symmetric classical r-matrices. We show, in particular, that in a general case in order for the Gaudin system in a magnetic field to be integrable, the corresponding magnetic field should be non-homogeneous.

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1. Introduction

There are two large classes of integrable spin chains. One of them is the celebrated Heisenberg spin chains (see review [1] and references therein) with the nearest-neighbors interaction. Another is the Gaudin spin chains [2] with a long-range interaction between spins in the chain. Known examples of the Gaudin spin chains are connected with skew-symmetric solutions of the classical Yang–Baxter equations [2, 3, 4, 6]. All such solutions are known, classified [7] and are exhausted by rational, trigonometric and elliptic solutions.

In our previous paper [8], we have constructed new integrable classical spin chains starting from general non-skew symmetric solutions of the generalized classical Yang–Baxter equations [9–11, 13, 14] with values in a semi-simple (reductive) Lie algebra $\mathfrak{g}$. In particular, we have constructed the second order in spin variables classical Hamiltonians of these models and showed that they are direct generalizations of the famous Gaudin Hamiltonians. In our paper [17], we have proved quantum integrability of the constructed systems, showing by the direct calculation, that the corresponding 'quantum' analogs of our classical generalized Gaudin Hamiltonians mutually commute. In the papers [17, 18], we have also constructed new non-skew-symmetric classical r-matrices and explicitly obtained many new examples of integrable quantum spin chains that generalize the Gaudin spin chains.

In the present paper, we investigate the problem of integrability of generalized quantum Gaudin Hamiltonians with the external magnetic field. For this purpose we, at first, explore the question of their classical integrability. We take into account that the Hamiltonians of the spin chains in a magnetic field are non-homogeneous, namely they consist of two parts: the quadratic in the spin variables part describing the interaction among the spins of the chain and linear in spin variables part describing the interaction of the spins of the chain with an external magnetic field. On the other hand, for classical one-top systems there exists the method that permits one to construct non-homogeneous integrable Hamiltonians, namely the so-called argument-shift method [19, 20]. In order to achieve our goal of constructing of classically integrable spin chains in a magnetic field we generalize the 'shift of the argument' method onto the case of an arbitrary r-matrix bracket and arbitrary Lax matrices possessing the such bracket. In the particular case of the 'many-poled' Lax matrices $L(u)$ with simple poles we obtain, after the 'generalized shift argument procedure', the needed classical Hamiltonians of the generalized Gaudin systems in an external magnetic field. The role of the 'external' magnetic field is played by some constant not depending on the dynamical variables element $c(\lambda) \in \mathfrak{g}(\lambda, \lambda^{-1})$). We call it the 'generalized shift element'. We show that in the case of the standard skew-symmetric matrix of Yang it coincides with an arbitrary spectral-parameter-independent element $c \in \mathfrak{g}$ from the usual 'argument-shift' method.

In a general case, classical integrability does not automatically imply quantum integrability due to the fact that commutativity of the classical integrals does not always imply the commutativity of the quantum ones. Nevertheless, by a direct calculation we also prove the commutativity of quantum analogs of the constructed classical Hamiltonians of the generalized Gaudin systems in an external magnetic field for all classical non-skew-symmetric non-dynamical r-matrices $r(\lambda, \mu)$ and all 'shift elements' $c(\lambda)$. The quantum-commuting Hamiltonians have the following explicit form:

$$\hat{H}^l_c = \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} \sum_{k\neq l}^{N} r^{\alpha\beta}(\nu_k, \nu_l)\hat{S}^k_\alpha \hat{S}^l_\beta + \frac{1}{2}\sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} r_0^{\alpha\beta}(\nu_l, \nu_l)\big(\hat{S}^l_\alpha \hat{S}^l_\beta + \hat{S}^l_\beta \hat{S}^l_\alpha\big) + \sum_{\alpha=1}^{\dim \mathfrak{g}} c^\alpha(\nu_l)\hat{S}^l_\alpha, \qquad (1)$$

where $\hat{S}^k_\alpha$ are the α-components of the generalized spin operators living in the site k ($k \in \overline{1, N}$) with the coordinate ν_k, $r^{\alpha\beta}(\lambda, \mu)$ are matrix elements of the r-matrix $r(\lambda, \mu)$ and $r_0^{\alpha\beta}(\lambda, \mu)$ are matrix elements of its regular part. It is necessary to note, that in a general case the 'shift element' $c(\lambda) = \sum_{\alpha=1}^{\dim \mathfrak{g}} c^\alpha(\lambda) X_\alpha$, playing the role of the external magnetic field, is λ-dependent and, hence, the external magnetic field is non-homogeneous.

In order to make our construction more concrete, in the present paper we explicitly construct admissible shift elements for several classes of classical r-matrices and corresponding Hamiltonians of spin chains in a magnetic field. In particular, we explicitly construct the constant 'shift elements' for the classical skew-symmetric r-matrices $r(\lambda - \mu)$, 'twisted' non-skew-symmetric r-matrices $r^\sigma(\lambda, \mu)$, where σ is an involutive automorphism [18] and 'anisotropic' non-skew-symmetric r-matrices $r^A(\lambda, \mu)$ [8, 32].

The structure of the present paper is the following. In section 2, we consider the case of classical integrable systems and introduce the 'generalized argument shift method'. In section 3, we consider the problem of a quantization of the obtained systems, prove commutativity of the generalized quantum Gaudin Hamiltonians in an external magnetic field and consider several classes of examples.

2. Classical case

2.1. Classical integrable systems and classical r-matrices

In this subsection, we will remind several basic facts [8, 10, 11] about general classical r-matrices with a spectral parameter and their relation to the theory of integrable systems.

We will use the following definition:

Definition 1. *The function of two complex variables $r(\lambda,\mu)$ with values in the tensor square of the algebra $\mathfrak{g}$ is called a classical r-matrix if it satisfies the 'generalized' or 'permuted' classical Yang–Baxter equation [11, 13, 14, 16]:*

$$[r_{12}(\lambda,\mu), r_{13}(\lambda,\nu)] = [r_{23}(\mu,\nu), r_{12}(\lambda,\mu)] - [r_{32}(\nu,\mu), r_{13}(\lambda,\nu)]. \tag{2}$$

Let X_α, $\alpha = \overline{1, \dim\mathfrak{g}}$ be some basis in $\mathfrak{g}$ with the commutation relations

$$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{\dim\mathfrak{g}} C^{\gamma}_{\alpha\beta} X_\gamma. \tag{3}$$

Then the corresponding r-matrix may be written as follows: $r(\lambda,\mu) = \sum_{\alpha\beta=1}^{\dim\mathfrak{g}} r^{\alpha\beta}(\lambda,\mu) X_\alpha \otimes X_\beta$.

Remark 1. If matrix $r(\lambda,\mu)$ is 'skew-symmetric', i.e. $r_{12}(\lambda,\mu) = -r_{21}(\mu,\lambda)$ equation (2) is reduced to the usual classical Yang–Baxter equation [7],

$$[r_{12}(\lambda,\mu), r_{13}(\lambda,\nu)] = [r_{23}(\mu,\nu), r_{12}(\lambda,\mu) + r_{13}(\lambda,\nu)]. \tag{4}$$

Due to the fact, that all solutions of the usual classical Yang–Baxter equation are skew [21] each solutions of (4) is also a solution of (2). The classical Yang–Baxter equation (4) may be obtained from the quantum Yang–Baxter equation [22–24]:

$$R_{12}(\lambda,\mu)R_{13}(\lambda,\nu)R_{23}(\mu,\nu) = R_{23}(\mu,\nu)R_{13}(\lambda,\nu)R_{12}(\lambda,\mu), \tag{5}$$

after the 'quasi-classical limit': $R_{12}(\lambda,\mu) = I + \eta r_{12}(\lambda,\mu) + o(\eta)$, where η is an additional parameter that label the solutions of (5). Note, that for general non-skew-symmetric r-matrices $r_{12}(\lambda,\mu)$ there is no analog of the quantum Yang–Baxter equation (5) and no quantum R-matrix $R_{12}(\lambda,\mu)$.

Remark 2. Note, that generalized classical Yang–Baxter equation possesses additional symmetry in comparison with the ordinary one, namely if $r_{12}(\lambda,\mu)$ is a solution of equation (2) then $r^f_{12}(\lambda,\mu) = f(\mu) r_{12}(\lambda,\mu)$ is also a solution of equation (2) for any scalar function $f(\mu)$. We consider r-matrices $r_{12}(\lambda,\mu)$ and $r^f_{12}(\lambda,\mu)$ to be equivalent.

Having the classical r-matrix $r(\lambda,\mu)$ one defines [11, 13, 14, 16] the following bracket:

$$\{L_1(\lambda), L_2(\mu)\} = [r_{12}(\lambda,\mu), L_1(\lambda)] - [r_{21}(\mu,\lambda), L_2(\mu)], \tag{6}$$

where $L_1(\lambda) = L(\lambda)\otimes 1$, $L_2(\mu) = 1\otimes L(\mu)$ and $L(\lambda) = \sum_{\alpha=1}^{\dim\mathfrak{g}} L^\alpha(\lambda) X_\alpha$.

It is evident that the so-defined bracket is skew. The Jacobi condition for the bracket (6) is provided by the fact, that $r(\lambda,\mu)$ satisfies (2). In a component form bracket (6) acquires the following explicit form:

$$\{L^\alpha(\lambda), L^\beta(\mu)\} = \sum_{\gamma,\delta=1}^{\dim\mathfrak{g}} \left(c^{\alpha}_{\gamma\delta} r^{\gamma\beta}(\lambda,\mu) L^\delta(\lambda) - c^{\beta}_{\gamma\delta} r^{\gamma\alpha}(\mu,\lambda) L^\delta(\mu) \right). \tag{7}$$

The following proposition follows from the explicit form of the Poisson bracket [11, 13, 14]:

Proposition 2.1. *Let $C^k(L)$, $L \in \mathfrak{g}^*$ be a Casimir function of order k of the algebra $\mathfrak{g}$. Then the functions $I^k(\lambda) = C^k(L(\lambda))$ are the generating functions of a commutative subalgebra in the Lie algebra of functions with respect to the brackets (6):*

$$\{I^k(\lambda), I^l(\mu)\} = 0.$$

Remark 3. In the case of matrix Lie algebras $\mathfrak{g}$ one may simply put $C^k(L) \equiv Tr L^k$.

Remark 4. The commutative integrals given by the generating functions $I^k(\lambda)$ can be written more explicitly as follows:

$$I_n^{r,i} = \mathrm{res}_{\lambda=\nu_i}(\lambda - \nu_i)^n C^r(L(\lambda)). \tag{8}$$

In what follows we will make an important 'regularity' assumption about behavior of the $r(\lambda, \mu)$ in a neighborhood of $\lambda = \nu_k, \mu = \nu_m$. In more details, we will assume that in the neighborhood of the points $\lambda = \nu_k, \mu = \nu_m, \forall k, m \in 1, N, k \neq m$ the function $r(\lambda, \mu)$ possesses the following decomposition:

$$r(\lambda, \mu) = \frac{\Omega}{(\lambda - \mu)} + r_0(\lambda, \mu), \tag{9}$$

where $r_0(\lambda, \mu)$ is a regular in a neighborhood of these points $\mathfrak{g}\otimes\mathfrak{g}$-valued function, $\Omega \in \mathfrak{g}\otimes\mathfrak{g}$ is the tensor Casimir: $\Omega = \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} g^{\alpha,\beta} X_\alpha \otimes X_\beta$, $g^{\alpha,\beta}$ is a nondegenerate invariant metric on $\mathfrak{g}$.

2.2. Classical r-matrices and 'shift of the argument' procedure

Let us describe how having Lax matrices that satisfy linear brackets (6) to obtain in the simple way the other Lax matrices satisfying linear brackets (6) with the same r-matrix.

The following proposition holds true:

Proposition 2.2. *Let $L(\lambda)$ be the Lax matrix with the Poisson bracket (6) and $c(\lambda)$ be a constant (i.e. not depending on dynamical variables) solution of the equation*

$$[r_{12}(\lambda, \mu), c_1(\lambda)] - [r_{21}(\mu, \lambda), c_2(\mu)] = 0. \tag{10}$$

Then the matrix $L^c(\lambda) = L(\lambda) + c(\lambda)$ also has Poisson bracket (6).

Proof. It follows from the linearity of the bracket (6) and the fact that the $\mathfrak{g}$-valued function of $\lambda c(\lambda)$ does not depend on dynamical variables and, hence

$$\{L_1(\lambda), c_2(\mu)\} = \{c_1(\lambda), L_2(\mu)\} = \{c_1(\lambda), c_2(\mu)\} = 0.$$

From this and equality (10) we derive that $\left\{L_1^c(\lambda), L_2^2(\mu)\right\} = \left[r_{12}(\lambda, \mu), L_1^c(\lambda)\right] - \left[r_{21}(\mu, \lambda), L_2^c(\mu)\right]$.

That proves the proposition. □

Remark 5. In the component form equality (10) is written as follows:

$$\sum_{\gamma,\delta=1}^{\dim \mathfrak{g}} \left(C^\alpha_{\gamma\delta} r^{\gamma\beta}(\lambda, \mu) c^\delta(\lambda) - C^\beta_{\gamma\delta} r^{\gamma\alpha}(\mu, \lambda) c^\delta(\mu)\right) = 0. \tag{11}$$

It will also be convenient to have a differential analogue of the condition (10). The following proposition holds true.

Proposition 2.3. *Let the classical r-matrix satisfies regularity condition (9). Then the c-matrix satisfying condition (10) as a function of λ satisfies the following differential equation:*

$$[(r_0)_{12}(\lambda,\lambda), c_1(\lambda)] - [(r_0)_{21}(\lambda,\lambda), c_2(\lambda)] = [\Omega_{12}, \partial_\lambda c_2(\lambda)], \tag{12}$$

or in the component form

$$\sum_{\delta=1}^{\dim \mathfrak{g}} c_\delta^{\beta\alpha}\partial_\lambda c^\delta(\lambda) = \sum_{\gamma,\delta=1}^{\dim \mathfrak{g}} \left(c_{\gamma\delta}^{\alpha} r_0^{\gamma\beta}(\lambda,\lambda) - c_{\gamma\delta}^{\beta} r_0^{\gamma\alpha}(\lambda,\lambda)\right)c^\delta(\lambda).$$

Definition 2. *We will call the algebra-valued function $c(\lambda)$ satisfying equations (10)–(12) 'the generalized shift element'.*

In the following section, we will consider several examples of the generalized shift elements for several classes of the skew-symmetric and non-skew-symmetric classical r-matrices.

Let us pass to a description of the 'generalized shift of the argument method' itself. Combining proposition 2.2 and proposition 2.1 we obtain the following proposition:

Proposition 2.4. *Let $C^k(L)$, $L \in \mathfrak{g}^*$ be a Casimir function of order k of the algebra $\mathfrak{g}$ and $c(\lambda)$ be a constant solution of equation (10). Then the functions*

$$I_c^k(\lambda) = C^k(L(\lambda) + c(\lambda))$$

are generating functions of the commutative subalgebra in the Lie algebra of functions of $L(\lambda)$ with respect to the brackets (6).

This proposition generalizes onto the case of the arbitrary r-matrix the so-called shift of the argument method [19]. In order to show this we will consider the following example:

Example 1. Let us consider the simplest classical rational r-matrix [7] of Yang,

$$r(\lambda,\mu) = \frac{\Omega}{(\lambda-\mu)}. \tag{13}$$

The simplest possible Lax operator corresponding to this Lax matrix has the following form:

$$L(\lambda) = \frac{1}{\lambda}\sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} g^{\alpha\beta} S_\alpha X_\beta.$$

In this case, $r_0(\lambda,\mu) = 0$ and equation (12) yields the condition $\partial_\lambda c(\lambda) = 0$, i.e. in this case a shift element does not depend on λ. Hence, we obtain the following generators of the Abelian with respect to the Lie–Poisson bracket on $\mathfrak{g}^*$ algebra:

$$I_c^k(\lambda) = \lambda^{-k} C^k(L + \lambda c),$$

where $L \equiv \lambda L(\lambda)$ and $c \in \mathfrak{g} = \mathfrak{g}^*$ is an arbitrary constant element. It is easy to see that (up to the multiplier λ^{-k}) these are exactly the generators of the Mishchenko–Fomenko algebra [19].

Remark 6. For the above example of the Mishchenko–Fomenko algebra there exists a Lie-theoretical interpretation of the constant shift element based on a loop algebra in 'homogeneous' grading [15]. More generally, there exists a special generalization of this construction [15], based on the loop algebras in other gradings (and classical r-matrices associated with them [16]). There also exists a Lie-theoretical interpretation of the corresponding 'shift elements' c [15]. In the present paper, we will not develop Lie-theoretical approach to the generalized shift elements $c(\lambda)$ corresponding to the general classical r-matrices $r(\lambda, \mu)$, showing that it is possible to find the shift elements with the help of the r-matrix formalism only, using either differential equation (12) or some additional symmetry conditions.

Now we will explain how the generalized shift elements are connected for the proportional r-matrices. The following proposition is proved by the direct verification:

Proposition 2.5. *Let $c(\lambda)$ be the generalized shift element for the r-matrix $r(\lambda, \mu)$. Then $c^f(\lambda) = f(\lambda)c(\lambda)$ is the generalized shift element for the r-matrix $r^f(\lambda, \mu) = f(\lambda)r(\lambda, \mu)$.*

Remark 7. Note that in the case of the skew-symmetric classical r-matrices the 'generalized shift of the argument procedure' is connected with a standard quantum-group technique [25]. Indeed, in this case condition (10) acquires the form

$$[r_{12}(\lambda, \mu), c_1(\lambda) + c_2(\mu)] = 0.$$

Its analogue for the case of the quantum R-matrix is the condition [25]

$$[R_{12}(\lambda, \mu), C_1(\lambda, \eta)C_2(\mu, \eta)] = 0,$$

where η is the additional parameter labeling solutions of QYBE such that $R_{12}(\lambda, \mu) = 1 + \eta r(\lambda, \mu) + o(\eta)$. From this it follows, in particular that $[r_{12}(\lambda, \mu), C_1(\lambda, 0)C_2(\mu, 0)] = 0$ and the matrices $C(\lambda, \eta)$ and $c(\lambda)$ are connected as follows: $c(\lambda) = C(\lambda, 0)^{-1}(\partial_\eta C(\lambda, \eta))|_{\eta=0}$. The quantum-group version of the 'shift of the argument procedure' is multiplicative $T(u) \to C(u, \eta)T(u)$ [25] (see also remark 8).

2.3. Classical spin chains

Now let us pass to a consideration of the concrete physical model we are interested in, namely for the generalized classical spin chains. For this purpose we will fix the concrete form of the Lax matrix $L(\lambda)$ satisfying tensorial brackets (6).

The following proposition holds true [8]:

Proposition 2.6. *Let S^k_α be the coordinate functions on $(\mathfrak{g}^{\oplus N})^*$ with a Lie–Poisson bracket:*

$$\left\{S^k_\alpha, S^l_\beta\right\} = \delta^{kl} C^\gamma_{\alpha,\beta} S^k_\gamma. \tag{14}$$

Then the Lax matrices of the following form:

$$L(\lambda) = \sum_{\alpha=1}^{\dim \mathfrak{g}} L_\alpha(\lambda) X^\alpha = \sum_{k=1}^{N} \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} S^k_\alpha r^{\alpha,\beta}(\nu_k, \lambda) X_\beta, \tag{15}$$

satisfy tensorial brackets (6).

Let us show that Lax matrix (15) corresponds to the Gaudin-type classical spin system. For this purpose we will obtain generalizations of the Gaudin Hamiltonians using the second-order generating function $I^2(L(\lambda))$. The following proposition is true [8]:

Proposition 2.7. *Let $r(\lambda, \mu)$ possesses the decomposition (9) in the neighborhood of the points $\lambda = \nu_k$, $\mu = \nu_m$, $k \neq m$. Then the Hamiltonians $H^k \equiv \frac{1}{2} I_0^{2,k}$ have the form*

$$H^k = \sum_{m\neq k}^{N} \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} r^{\alpha,\beta}(\nu_m, \nu_k) S^m_\alpha S^k_\beta + \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} r_0^{\alpha,\beta}(\nu_k, \nu_k) S^k_\alpha S^k_\beta. \tag{16}$$

The Hamiltonian H_c^k may be interpreted as an energy of the spinning particle living at the site k that interact with the spins living at the other sites. The set of these Hamiltonians for different k are generalizations of the well-known Gaudin Hamiltonians [2]. Indeed, in the case of the skew-symmetric r-matrices $r(\lambda, \mu)$ we have that $r_0^{\alpha,\beta}(\nu_k, \nu_k) = -r_0^{\beta\alpha}(\nu_k, \nu_k)$, the second sum in formula (16) vanishes and Hamiltonians (16) are reduced to the usual classical Gaudin Hamiltonians.

2.4. Classical spin chain in a magnetic field

Let us now pass to a consideration of the main example of the present paper, namely generalized Gaudin systems in a magnetic field. Let us now obtain the corresponding Hamiltonians and Lax matrix. For this purpose we will apply the 'shift of the argument' procedure described in the previous subsection. The corresponding 'shifted' Lax matrix has the following form:

$$L^c(\lambda) = L(\lambda) + c(\lambda) = \sum_{\beta=1}^{\dim\mathfrak{g}} L_c^\beta(\lambda) X_\beta = \sum_{k=1}^{N} \sum_{\alpha,\beta=1}^{\dim\mathfrak{g}} S_\alpha^k r^{\alpha,\beta}(\nu_k, \lambda) X_\beta + \sum_{\alpha=1}^{\dim\mathfrak{g}} c^\alpha(\lambda) X_\alpha, \tag{17}$$

where $c(\lambda) = \sum_{\alpha=1}^{\dim\mathfrak{g}} c^\alpha(\lambda) X_\alpha$ is the shift element for the corresponding r-matrix $r(\lambda, \mu)$.

Let us consider the corresponding second-order Hamiltonians of the classical spin chain with the above general shift of the argument. The direct verification gives the following:

Proposition 2.8. *Let the shift element $c(\lambda)$ has no poles at the points of the chain $\lambda = \nu_k$. Then the Hamiltonians $H_c^k \equiv \frac{1}{2} I_0^{2,k}(L^c(\lambda))$ have the following explicit form:*

$$H_c^k = H^k + h^k = \left(\sum_{m\neq k}^{N} \sum_{\alpha,\beta=1}^{\dim\mathfrak{g}} r^{\alpha,\beta}(\nu_m, \nu_k) S_\alpha^m S_\beta^k + \sum_{\alpha,\beta=1}^{\dim\mathfrak{g}} r_0^{\alpha,\beta}(\nu_k, \nu_k) S_\alpha^k S_\beta^k \right) + \sum_{\alpha=1}^{\dim\mathfrak{g}} c^\beta(\nu_k) S_\beta^k. \tag{18}$$

The Hamiltonian H_c^k may be interpreted as an energy of the spinning particle living at the site k that interacts with spins living at the other sites and with the external non-homogeneous magnetic field $c(\lambda)$ with values $c(\nu_k)$ in the points $\lambda = \nu_k$. In the following section, we will consider the problem of a quantization of the obtained classical Hamiltonians and produce several explicit examples of the generalized shift elements and generalized quantum Gaudin models in a magnetic field.

3. Integrable quantum systems and classical *r*-matrices

3.1. General case

Let us consider the quantization of the classical integrable systems described in the previous subsection. In the case of the linear Poisson bracket (6) it is achieved by the substitution of the Lax matrix $L(\lambda) = \sum_{\alpha=1}^{\dim\mathfrak{g}} L^\alpha(\lambda) X_\alpha$ with classical dynamical variables coefficients by the operator-valued Lax matrix $\hat{L}(\lambda) = \sum_{\alpha=1}^{\dim\mathfrak{g}} \hat{L}^\alpha(\lambda) X_\alpha$ with quantum operators coefficients acting in the corresponding Hilbert space $\mathcal{H}$ of quantum states, such that the following commutation relation holds true:

$$[\hat{L}_1(\lambda), \hat{L}_2(\mu)] = [r_{12}(\lambda, \mu), \hat{L}_1(\lambda)] - [r_{21}(\mu, \lambda), \hat{L}_2(\mu)], \tag{19}$$

where $\hat{L}_1(\lambda) = \hat{L}(\lambda) \otimes 1$, $\hat{L}_2(\mu) = 1 \otimes \hat{L}(\mu)$.

Remark 8. In the case of the skew-symmetric r-matrices $r_{12}(\lambda,\mu) = -r_{21}(\mu,\lambda)$ bracket (19) acquires standard Sklyanin form [22]

$$[L_1(\lambda), L_2(\mu)] = [r_{12}(\lambda,\mu), L_1(\lambda) + L_2(\mu)]. \tag{20}$$

This Lie bracket can be obtained as a 'quasi-classical limit' of the FRT relation [22–24]

$$R_{12}(\lambda,\mu)T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)R_{12}(\lambda,\mu), \tag{21}$$

where $T(\lambda) = I + \eta L(\lambda) + o(\eta)$.

Note that for the Lax operators associated with the general non-skew-symmetric r-matrices $r_{12}(\lambda,\mu)$ there is no analog of the quantum-group relations (21).

Let us pass to the problem of the construction of quantum analogs of classical integrals. It is of possible to define generators $I^k(\hat{L}(\lambda))$ in a quantum case in the same way as in a classical case: $\hat{I}^k(\lambda) = C^k(\hat{L}(\lambda))$. Nevertheless, such a naive definition may not lead to a quantum integrable system because in a general case

$$[\hat{I}^k(\lambda), \hat{I}^l(\mu)] \neq \widehat{\{I^k(\lambda), I^l(\mu)\}} = 0.$$

This is so, because our quantization is a homomorphism from a Lie algebra of functions with the Poisson bracket to a Lie algebra of operators only on the subalgebra of functions linear in the basic dynamical variables $\hat{L}^\alpha(u)$, while $\hat{I}^k(\lambda)$ is polynomial in $\hat{L}^\alpha(u)$.

The problem of a construction of the commuting analogs of the classical generating functions $I^k(\lambda)$ is very complicated (see [26]). It was solved for general classical r-matrices only in the case $\mathfrak{g} = so(3)$ (see [31]). For the case $\mathfrak{g} = gl(n)$ it was solved (half-explicitly) only for the classical r-matrix of Yang [27] (see also [28]). Nevertheless, in the following subsection we will show by a direct calculation, that both generalized Gaudin Hamiltonians and generalized Gaudin Hamiltonians in a magnetic field stay commutative in a quantum case for an arbitrary classical r-matrix and an arbitrary Lie algebra $\mathfrak{g}$.

3.2. Case of quantum spin chains

Let $\hat{S}^i_\alpha$, $\alpha = 1, \dim\mathfrak{g}$, $i = 1, N$ be linear operators in some Hilbert space that span Lie algebra isomorphic to $\mathfrak{g}^{\oplus N}$ with the commutation relations

$$\left[\hat{S}^i_\alpha, \hat{S}^j_\beta\right] = \delta^{ij} \sum_{\gamma=1}^{\dim\mathfrak{g}} C^\gamma_{\alpha\beta} \hat{S}^j_\gamma. \tag{22}$$

It is evident that brackets (22) are the quantization of the Lie–Poisson brackets (14).

We will consider operators $\hat{S}_\alpha$ to be the α components of the 'generalized spin operator'. The operators $\hat{S}^i_\alpha$ could be interpreted as the α component of the generalized spin operator living at the i-site of the generalized spin chain.

It is possible to introduce the following 'quantum Lax operator' [8, 17]:

$$\hat{L}(\lambda) = \sum_{\beta=1}^{\dim\mathfrak{g}} \hat{L}^\beta(\lambda) X_\beta \equiv \sum_{k=1}^{N} \sum_{\alpha,\beta=1}^{\dim\mathfrak{g}} r^{\alpha,\beta}(\nu_k,\lambda) \hat{S}^k_\alpha X_\beta. \tag{23}$$

It is easy to show that it satisfies a linear r-matrix algebra

$$[\hat{L}_1(\lambda), \hat{L}_2(\mu)] = [r_{12}(\lambda,\mu), \hat{L}_1(\lambda)] - [r_{21}(\mu,\lambda), \hat{L}_2(\mu)], \tag{24}$$

i.e. is, indeed, a quantization of the classical Lax operator (15).

The following theorem is proved by the long and tedious calculations using generalized classical Yang–Baxter equations and some derived identities [17]:

Theorem 3.1. *Let $r(\lambda,\mu)$ be the classical r-matrix and $r_0(\lambda,\mu)$ its regular part. Then*

(i) The Hamiltonians $\hat{H}^l = 1/2\,\mathrm{res}_{\lambda=\nu_l} I^2(\lambda)$, where $I^2(\lambda) = \sum_{\alpha,\beta=1}^{\dim\mathfrak{g}} g_{\alpha\beta}\hat{L}^\alpha(\lambda)\hat{L}^\beta(\lambda)$ have the following explicit form:

$$\hat{H}^l = \sum_{\alpha,\beta=1}^{\dim\mathfrak{g}}\sum_{k\neq l}^{N} r^{\alpha\beta}(\nu_k,\nu_l)\hat{S}^k_\alpha\hat{S}^l_\beta + \frac{1}{2}\sum_{\alpha,\beta=1}^{\dim\mathfrak{g}} r_0^{\alpha\beta}(\nu_l,\nu_l)\big(\hat{S}^l_\alpha\hat{S}^l_\beta + \hat{S}^l_\beta\hat{S}^l_\alpha\big). \tag{25}$$

(ii) The Hamiltonians $\hat{H}^l, l\in\overline{1,N}$ form a commutative family in the universal enveloping algebra $\mathfrak{A}(\mathfrak{g}^{\oplus N})$ and its representations.

Remark 9. In the case of classical matrix Lie algebras Hamiltonians $\hat{H}^l$ may be written in the more simple form

$$\hat{H}^l = \tfrac{1}{2}\,\mathrm{res}_{\lambda=\nu_l}\,\mathrm{tr}(\hat{L}(\lambda))^2. \tag{26}$$

Remark 10. Note that the second 'self-action' term in the Hamiltonian (25) is important for the commutativity of the constructed generalized Gaudin Hamiltonians, i.e. for the integrability of these systems. It compensates non-skew-symmetry of the general classical r-matrix, entered into the first 'Gaudin-like' term in our Hamiltonian (see [17]).

Let us now analyze the equivalence among the generalized Gaudin systems. The following proposition holds true:

Proposition 9. *Let $\hat{H}^l$ be generalized Gaudin Hamiltonians constructed with the help of the classical r-matrix $r(\lambda,\mu)$ possessing decomposition (9). Then generalized Gaudin Hamiltonians corresponding to the classical r-matrix $r^f(\lambda,\mu) = f(\mu)r(\lambda,\mu)$ have the following form:*

$$\hat{H}^l_f = f(\nu_l)\hat{H}^l - \tfrac{1}{2}f'(\nu_l)\hat{C}^l, \tag{27}$$

where $\hat{C}^l = \sum_{\alpha,\beta=1}^{\dim\mathfrak{g}} g^{\alpha\beta}\hat{S}^l_\alpha\hat{S}^l_\beta$ are second-order Casimir operators on $\mathfrak{g}^{\oplus N}$,

Proof. In order to prove the proposition, we have to construct decomposition (9) of the r-matrix $r^f(\lambda,\mu) = f(\mu)r(\lambda,\mu)$. For this purpose we will make substitution of the local spectral parameters λ and μ:

$$\lambda = F(u),\qquad \mu = F(v),\qquad \text{where}\quad F'(u) = f(F(u)).$$

Using the regularity property of the r-matrix $r(\lambda,\mu)$ we will have

$$f(\mu)r(\lambda,\mu) = f(\mu)\frac{\Omega}{(\lambda-\mu)} + f(\mu)r_0(\lambda,\mu) = \frac{f(F(v))\Omega}{(F(u)-F(v))} + f(F(v))r_0(F(u),F(v))$$
$$= \frac{\Omega}{(u-v)} + \left(f(F(v))r_0(F(u),F(v)) - \frac{F''(v)}{2(F'(v))}\Omega\right) + o(u-v),$$

where we have used the decomposition of $(F(u)-F(v))^{-1}$ in the Laurent power series in $(u-v)$. Using the rule of the differentiation and our definition of the function F $(F'(u) = f(F(u)))$ we obtain that $F''(v) = f'(\mu(v))f(\mu(v))$ (where in the left-hand side differentiation is implied with respect to the parameter v and in the right-hand side—with respect to the parameter μ). Using this we finally obtain that

$$r_0^f(\mu(v),\mu(v)) \equiv f(\mu(v))r_0(\mu(v),\mu(v)) - \tfrac{1}{2}f'(\mu)\Omega.$$

Defining with the help of $r^f(\mu(u),\mu(v))$ and $r_0^f(\mu(v),\mu(v))$ the generalized Gaudin Hamiltonians $\hat{H}^k_f$ we come to formula (27). Proposition is proved. □

Remark 11. From this proposition it follows that the generalized Gaudin systems constructed with the help of the equivalent r-matrices $r(\lambda,\mu)$ and $r^f(\lambda,\mu)$ are equivalent. In particular, the generalized Gaudin systems constructed with the help of the r-matrices $r^f(\lambda,\mu)$, where r-matrix $r(\lambda,\mu)$ is skew-symmetric are equivalent to the ordinary Gaudin systems.

Remark 12. Note that the ordinary Gaudin Hamiltonians can be obtained using the quantum-group technique and the above mentioned quasi-classical limit. Their commutativity may also be proved in such a way. This is not true in the case of general non-skew-symmetric r-matrices because, as it was mentioned above, in this general case there is no quantum-group structure. Moreover, even in the standard case of skew-symmetric r-matrices the task of obtaining of Gaudin systems starting from quantum groups is non-trivial [5] and it is more natural to obtain Gaudin Hamiltonians in the classical r-matrix setting.

3.3. Case of quantum spin chains in a magnetic field

Now we are interested in obtaining of the quantum analogs of the Hamiltonians (18) that will be the Hamiltonians of a system of N spins in a magnetic field and in showing their commutativity. Let us consider the corresponding 'shifted' quantum Lax matrix

$$\hat{L}^c(\lambda) = \hat{L}(\lambda) + c(\lambda) = \sum_{\beta=1}^{\dim\mathfrak{g}} \hat{L}_c^\beta(\lambda) X_\beta \equiv \sum_{k=1}^{N}\sum_{\alpha,\beta=1}^{\dim\mathfrak{g}} r^{\alpha,\beta}(\nu_k,\lambda)\hat{S}_\alpha^k X_\beta + \sum_{\alpha=1}^{\dim\mathfrak{g}} c^\alpha(\lambda) X_\alpha. \tag{28}$$

(The generalized shift element $c(\lambda)$ is a constant c-numbered function of λ both in classical and quantum cases.)

The following analogue of theorem (3.1) holds true:

Theorem 3.2. *Let $r(\lambda,\mu)$ be the classical r-matrix and $r_0(\lambda,\mu)$ its regular part. Let the shift element $c(\lambda)$ has no poles at the points of the chain $\lambda = \nu_k$. Then*

(i) The Hamiltonians $\hat{H}_c^l = \frac{1}{2}\operatorname{res}_{\lambda=\nu_l} I_c^2(\lambda)$, where $I_c^2(\lambda) = \sum_{\alpha,\beta=1}^{\dim\mathfrak{g}} g_{\alpha\beta}\hat{L}_c^\alpha(\lambda)\hat{L}_c^\beta(\lambda)$ *have the following explicit form:*

$$\hat{H}_c^l = \sum_{\alpha,\beta=1}^{\dim\mathfrak{g}}\sum_{k\neq l}^{N} r^{\alpha\beta}(\nu_k,\nu_l)\hat{S}_\alpha^k\hat{S}_\beta^l + \frac{1}{2}\sum_{\alpha,\beta=1}^{\dim\mathfrak{g}} r_0^{\alpha\beta}(\nu_l,\nu_l)\big(\hat{S}_\alpha^l\hat{S}_\beta^l + \hat{S}_\beta^l\hat{S}_\alpha^l\big) + \sum_{\alpha=1}^{\dim\mathfrak{g}} c^\alpha(\nu_l)\hat{S}_\alpha^l. \tag{29}$$

(ii) The Hamiltonians $\hat{H}_c^l, l \in \overline{1,N}$ form a commutative family in the universal enveloping algebra $\mathfrak{A}(\mathfrak{g}^{\oplus N})$ and its representations.

Proof. Item (i) of the theorem is checked by the direct verification. Let us prove item (ii). Due to the fact that in general in a quantum case there is no proof of a commutativity of generating functions $I_c^2(\lambda)$ we will prove this item of the theorem directly, showing that $\big[\hat{H}_c^k,\hat{H}_c^l\big] = 0$. For this purpose we will use that $\hat{H}_c^k = \hat{H}^k + \hat{h}^k$, where $\hat{H}^k$ is given by formula (25) and $\hat{h}^k \equiv \sum_{\alpha=1}^{\dim\mathfrak{g}} c^\alpha(\nu_l)\hat{S}_\alpha^l$. We will, moreover, use for $\hat{H}^l$ the following decomposition: $\hat{H}^l = \hat{H}_0^l + \hat{H}_1^l$, where $\hat{H}_0^l \equiv \sum_{\alpha,\beta=1}^{\dim\mathfrak{g}}\sum_{k\neq l}^{N} r^{\alpha\beta}(\nu_k,\nu_l)\hat{S}_\alpha^k\hat{S}_\beta^l$, $\hat{H}_1^l = \frac{1}{2}\sum_{\alpha,\beta=1}^{\dim\mathfrak{g}} r_0^{\alpha\beta}(\nu_l,\nu_l)\big(\hat{S}_\alpha^l\hat{S}_\beta^l + \hat{S}_\beta^l\hat{S}_\alpha^l\big)$. We will have $(k\neq l)$:

$$\big[\hat{H}_c^k,\hat{H}_c^l\big] = \big[\hat{H}_0^k + \hat{H}_1^k + \hat{h}^k, \hat{H}_0^l + \hat{H}_1^l + \hat{h}^l\big] = \big[\hat{H}_0^k,\hat{h}^l\big] + \big[\hat{h}^k,\hat{H}_0^l\big],$$

where we have used that $[\hat{H}^k,\hat{H}^l] = 0$ by the virtue of theorem (3.1) and $\big[\hat{H}_1^k,\hat{h}^l\big] = \big[\hat{h}^k,\hat{H}_1^l\big] = [\hat{h}^k,\hat{h}^l] = 0$, due to the fact that $k\neq l$ and spin operators living at the different sites commute with each other. At last we will have

$$\left[\hat{H}_0^k, \hat{h}^l\right] + \left[\hat{h}^k, \hat{H}_0^l\right] = \sum_{\beta,\gamma,\delta=1}^{\dim \mathfrak{g}} \sum_{m\neq k}^{N} \left[r^{\gamma\beta}(\nu_m, \nu_k)\hat{S}_\gamma^m \hat{S}_\beta^k, c^\delta(\nu_l)\hat{S}_\delta^l\right] - \left[r^{\gamma\beta}(\nu_m, \nu_l)\hat{S}_\gamma^m \hat{S}_\beta^l, c^\delta(\nu_k)\hat{S}_\delta^k\right]$$

$$= \sum_{\alpha,\beta,\gamma,\delta=1}^{\dim \mathfrak{g}} \left(r^{\gamma\beta}(\nu_l, \nu_k)c^\delta(\nu_l)C_{\gamma\delta}^\alpha \hat{S}_\beta^k \hat{S}_\alpha^l - r^{\gamma\beta}(\nu_k, \nu_l)c^\delta(\nu_k)C_{\gamma\delta}^\alpha \hat{S}_\beta^l \hat{S}_\alpha^k\right)$$

$$= \sum_{\alpha,\beta,\gamma,\delta=1}^{\dim \mathfrak{g}} \left(r^{\gamma\beta}(\nu_l, \nu_k)c^\delta(\nu_l)C_{\gamma\delta}^\alpha - r^{\gamma\alpha}(\nu_k, \nu_l)c^\delta(\nu_k)C_{\gamma\delta}^\beta\right)\hat{S}_\alpha^l \hat{S}_\beta^k = 0,$$

where we have used that $c^\delta(\nu_k)$ are the components of the 'shift element' that satisfy condition (11). Hence $\left[\hat{H}_c^k, \hat{H}_c^l\right] = 0$.

Theorem is proved. □

Remark 13. Let us note, that in the case of skew-symmetric classical r-matrices and ordinary Gaudin systems in a magnetic field the corresponding Hamiltonians may be obtained using the standard quantum-group technique. Magnetic field (shift element) in such Hamiltonians is constructed as in remark 7.

Let us briefly comment on the equivalences among the constructed systems. From propositions 15 and 24 follows the following corollary:

Corollary 3.1. *Let $\hat{H}_c^l$ be the generalized Gaudin Hamiltonians in a magnetic field constructed with the help of the classical r-matrix $r(\lambda, \mu)$ and $\hat{C}^l$ be the second-order Casimir operators on $\mathfrak{g}^{\oplus N}$. Then the generalized Gaudin Hamiltonians in magnetic field corresponding to the classical r-matrix $r^f(\lambda, \mu) = f(\mu)r(\lambda, \mu)$ have the form*

$$\hat{H}_{c,f}^l = f(\nu_l)\hat{H}_c^l - \tfrac{1}{2}f'(\nu_l)\hat{C}^l, \tag{30}$$

i.e. the generalized Gaudin systems in a magnetic field constructed with the help of the equivalent classical r-matrix $r(\lambda, \mu)$ and $r^f(\lambda, \mu)$ are equivalent.

Theorem 3.2 proves a quantum integrability of the 'generalized Gaudin systems' in the external magnetic field, the form of which is determined by the 'shift element' $c(\lambda)$. In order to make our construction more concrete we will explicitly consider several examples.

3.4. Examples of the generalized Gaudin systems in magnetic field

3.4.1. Case of skew-symmetric r-matrices. Let us consider the case of the classical skew-symmetric r-matrices: $r_{12}(\lambda, \mu) = -r_{21}(\mu, \lambda)$. The corresponding Gaudin Hamiltonian in a magnetic field acquires the more standard form

$$\hat{H}_c^l = \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} \sum_{k\neq l}^{N} r^{\alpha\beta}(\nu_k, \nu_l)\hat{S}_\alpha^k \hat{S}_\beta^l + \sum_{\alpha=1}^{\dim \mathfrak{g}_0} c^\alpha(\nu_k)\hat{S}_\alpha^l, \tag{31}$$

where the shift element $c(\lambda)$ satisfies the condition

$$[r_{12}(\lambda, \mu), c_1(\lambda) + c_2(\mu)] = 0.$$

In order to find explicitly some solutions of this equations we will assume that the skew-symmetric r-matrix possesses a symmetry with respect to some finite-dimensional Lie group $G_0 \subset G$, where G is a Lie group of the Lie algebra $\mathfrak{g}$, i.e.,

$$Ad_g \otimes Ad_g r(\lambda, \mu) = r(\lambda, \mu), \qquad \forall g \in G_0.$$

In such a case, it is easy to show that

$$[r(\lambda,\mu), X\otimes 1 + 1\otimes X] = 0, \qquad \forall X \in \mathfrak{g}_0,$$

where $\mathfrak{g}_0$ is the Lie algebra of the Lie group G_0. Hence $c \equiv \sum_{\alpha=1}^{\dim \mathfrak{g}_0} c^\alpha X_\alpha$ is the correctly defined 'homogeneous' ($c(\nu_k) = c$) shift element.

Let us consider two examples of the constant shift elements corresponding to two r-matrices with a different symmetry:

Example 2. Let $r_{12}(\lambda,\mu) = \frac{\Omega_{12}}{\lambda-\mu}$ be the standard r-matrix of Yang. Then $[r(\lambda,\mu), X\otimes 1 + 1\otimes X] = (\lambda-\mu)^{-1}[\Omega_{12}, X\otimes 1 + 1\otimes X] = 0, \forall X \in \mathfrak{g}$ by the very definition of the tensor Casimir Ω_{12}. Hence any constant element of $\mathfrak{g}$ may be taken to be the shift element. The corresponding Gaudin Hamiltonians in a magnetic field (31) are standard ones [4, 29, 28].

Example 3. Let $\mathfrak{g}$ be a simple Lie algebra of the rank r. Let $r_{12}(\lambda,\mu)$ be the standard trigonometric r-matrix [7]:

$$r(\lambda,\mu) = \frac{\mu^h+\lambda^h}{2(\lambda^h-\mu^h)}\left(\sum_{i=1}^{\dim\mathfrak{h}} H_i\otimes H_i\right) + \frac{\mu^h}{(\lambda^h-\mu^h)}\left(\sum_{\alpha\in\Delta}\lambda^{l(\alpha)}\mu^{-l(\alpha)}E_\alpha\otimes E_{-\alpha}\right), \tag{32}$$

where Δ is a set of roots of the Lie algebra $\mathfrak{g}$, $l(\alpha)$ is the height of root α, E_α is a basic vector of the corresponding root space, H_i is the basis vector of the Cartan subalgebra $\mathfrak{h} \equiv \mathfrak{g}_{\bar{0}}$ with the normalization: $(E_\alpha, E_{-\alpha}) = 1, (H_i, H_i) = 1$ and h is the Coxeter number of $\mathfrak{g}$. A replacement of the spectral parameters $\lambda = e^u, \mu = e^v$ reduces this r-matrix to the form depending on the difference of spectral parameters $u - v$. It is easy to see that r-matrix (32) is invariant with respect to the Cartan subgroup H. Hence the arbitrary element of the Cartan subalgebra

$$c = \sum_{i=1}^{\dim\mathfrak{h}} c_i H_i$$

is a correct shift element for the standard trigonometric r-matrix. The corresponding Gaudin Hamiltonians in magnetic field (31) were considered, for example, in [30].

Remark 14. The constructed shift elements for the skew-symmetric r-matrices are independent of a spectral parameter. Below we will give an example of spectral-parameter dependent (non-homogeneous) shift element for the case of non-skew symmetric r-matrices.

3.4.2. Case of 'σ-twisted' r-matrices $r^\sigma_{12}(\lambda,\mu)$. Let us consider the case of a non-skew-symmetric classical r-matrix $r(\lambda,\mu)$ obtained from a skew-symmetric one by the 'twisting' with the help of the some involutive automorphism σ [18]. In more details, let σ be an involutive automorphism of $\mathfrak{g}$, i.e. $\sigma^2 = 1$ and $\sigma([X,Y]) = [\sigma(X),\sigma(Y)], \forall X, Y \in \mathfrak{g}$. We will use the following notations: $\sigma_1 = \sigma\otimes 1, \sigma_2 = 1\otimes\sigma$, etc. Let $\widetilde{\sigma}$ be the lift of σ onto the algebra of $\mathfrak{g}$-valued functions given by the formula: $\widetilde{\sigma}X(\lambda) = \sigma X(-\lambda)$, and on the level of their tensor product by the formula $(\widetilde{\sigma_1}\widetilde{\sigma_2})r_{12}(\lambda,\mu) = (\sigma_1\sigma_2)r_{12}(-\lambda,-\mu)$.

Let $r_{12}(\lambda-\mu)$ be a skew-symmetric r-matrix which is anti-invariant with respect to the action of the automorphism $\widetilde{\sigma}$:

$$(\widetilde{\sigma_1\sigma_2})r_{12}(\lambda-\mu) = -r_{12}(\lambda-\mu). \tag{33}$$

Then, as it was shown in [18], in such a case the function

$$r^\sigma_{12}(\lambda,\mu) = (1-\widetilde{\sigma_2})r_{12}(\lambda-\mu) = r_{12}(\lambda-\mu) - \sigma_2 r_{12}(\lambda+\mu). \tag{34}$$

is a non-skew symmetric classical r-matrix.

In this case, it is easy to show [18] that $\left(r_0^{\sigma}\right)_{12}(\lambda,\lambda)=\sigma_2 r_{12}(2\lambda)$ and the 'twisted' Gaudin Hamiltonians in the magnetic field corresponding to this r-matrix have the form

$$\hat{H}^l=\sum_{k=1,k\neq l}^{N}\sum_{\alpha,\beta=1}^{\dim\mathfrak{g}}(r^{\alpha\beta}(\nu_k-\nu_l)-(\sigma_2 r)^{\alpha\beta}(\nu_k+\nu_l))\hat{S}^k_\alpha\hat{S}^l_\beta - \sum_{\alpha,\beta=1}^{\dim\mathfrak{g}}(\sigma_2 r)^{\alpha\beta}(2\nu_l)\hat{S}^l_\alpha\hat{S}^l_\beta+\sum_{\alpha=1}^{\dim\mathfrak{g}}c^{\alpha}(\nu_l)\hat{S}^l_\alpha. \tag{35}$$

Let us explicitly describe the shift elements $c(\lambda)$ for this case. Due to the fact that $r^{\sigma}_{12}(\lambda,\mu)$ is constructed from the skew-symmetric r-matrix $r_{12}(\lambda-\mu)$, the corresponding shift elements will be also constructed from the shift elements of the skew-symmetric r-matrix. The following proposition holds true:

Proposition 3.2. *Let $r_{12}(\lambda-\mu)$ be a skew-symmetric classical r-matrix anti-invariant with respect to an automorphism σ of the second order. Let $c(\lambda)\in\mathfrak{g}(\lambda^{-1},\lambda))$ be some shift element for $r(\lambda-\mu)$, i.e. $[r(\lambda-\mu),c_1(\lambda)+c_2(\mu)]=0$, which is anti-invariant under the action of σ: $\widetilde{\sigma}(c(\lambda))=\sigma(c(-\lambda))=-c(\lambda)$. Then $c(\lambda)$ is also a shift element for the r-matrix $r^{\sigma}_{12}(\lambda,\mu)$:*

$$\left[r^{\sigma}_{12}(\lambda,\mu),c_1(\lambda)\right]-\left[r^{\sigma}_{21}(\mu,\lambda),c_2(\mu)\right]=0.$$

Proof. By direct calculation, using the fact, that in the cases of the 'σ anti-invariant' r-matrices $r(\lambda-\mu)$ there exists the following symmetry property [18]: $\sigma_2 r_{12}(\lambda+\mu)=\sigma_1 r_{21}(\lambda+\mu)$ we obtain:

$$\begin{aligned}\left[r^{\sigma}_{12}(\lambda,\mu),c_1(\lambda)\right]-\left[r^{\sigma}_{21}(\mu,\lambda),c_2(\mu)\right]&=[r_{12}(\lambda-\mu)-\sigma_2 r_{12}(\lambda+\mu),c_1(\lambda)]\\&\quad-[r_{21}(\mu-\lambda)-\sigma_1 r_{21}(\lambda+\mu),c_2(\mu)]\\&=[r_{12}(\lambda-\mu),c_1(\lambda)+c_2(\mu)]-[\widetilde{\sigma_2}r_{12}(\lambda-\mu),c_1(\lambda)-c_2(\mu)]\\&=(1-\widetilde{\sigma_2})[r_{12}(\lambda-\mu),c_1(\lambda)+c_2(\mu)]=0,\end{aligned}$$

where we have used the skew-symmetry of the r-matrix $r_{12}(\lambda-\mu)$, the fact that $\widetilde{\sigma_2}c_1(\lambda)=c_1(\lambda)$ and that by the conditions of the proposition $[r_{12}(\lambda-\mu),c_1(\lambda)+c_2(\mu)]=0$, $\widetilde{\sigma_2}c_2(\mu)=-c_2(\mu)$ and $\widetilde{\sigma_2}$ is an automorphism.

Proposition is proved. □

Example 4. Let us consider the partial case of the above proposition when $\sigma=id$. In this case, we have that the anti-invariance condition is equivalent to the following parity condition: $r_{12}(\lambda-\mu)=-r_{12}(\mu-\lambda)$. The corresponding 'twisted' non-skew-symmetric r-matrix has the simple form

$$r^{\sigma}_{12}(\lambda,\mu)=r_{12}(\lambda-\mu)-r_{12}(\lambda+\mu).$$

In this case, any odd shift element $c(\lambda)=-c(-\lambda)$ for the skew-symmetric r-matrix $r(\lambda-\mu)$ is also a shift element for the non-skew-symmetric r-matrix $r^{\sigma}_{12}(\lambda,\mu)$.

Example 5. Let us consider the case of an arbitrary involutive automorphism σ, and the constant shift elements $c(\lambda)\equiv c$. In this case for such shift elements one can take arbitrary constant shift elements constructed in the previous subsections. In order for the element c to be also a shift element for the r-matrix $r^{\sigma}(\lambda,\mu)$ one has to require that $\sigma(c)=-c$. For example, when $r(\lambda-\mu)$ is the classical r-matrix of Yang, c is an arbitrary element of $\mathfrak{g}$ and the condition $\sigma(c)=-c$ is equivalent to the condition that $c\in\mathfrak{g}_{\bar{1}}$, where $\mathfrak{g}=\mathfrak{g}_{\bar{0}}+\mathfrak{g}_{\bar{1}}$ is Z_2 grading of $\mathfrak{g}$ corresponding to the automorphism σ.

3.4.3. Case of anisotropic non-skew-symmetric r-matrices $r^A(\lambda,\mu)$. Let us consider the other class of non-skew-symmetric r-matrices which will explicitly provides us an example of 'non-homogeneous shift element'.

For this purpose we will remind the construction of 'anisotropic' r-matrices in the special case $\mathfrak{g} = gl(n)$ [8, 17, 32]. In more details, let $a_i, i \in \overline{1,n}$ be some complex numbers (anisotropy parameters). Let X_{ij} be the standard basis of $gl(n) : (X_{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$ with the commutation relations

$$[X_{ij}, X_{kl}] = \delta_{kj}X_{il} - \delta_{il}X_{kj}.$$

It is possible to show [17, 32] that the following element of $\mathfrak{g}\otimes\mathfrak{g}$,

$$r^A(\lambda,\mu) = \frac{1}{(\lambda-\mu)}\sum_{i,j=1}^{n}\frac{(\lambda-a_i)}{(\mu-a_i)}X_{ij}\otimes X_{ji}, \tag{36}$$

satisfies equation (2).

Remark 15. As it was shown in [32] the rational r-matrices (36) are gauge equivalent to a very special 'hyperelliptic' r-matrices discovered in [8].

Direct calculation gives [17] the following expression for the 'regular part' $r_0^A(\lambda,\lambda)$:

$$r_0^A(\lambda,\lambda) = \sum_{i,j=1}^{n}\frac{1}{(\lambda-a_i)}X_{ij}\otimes X_{ji}. \tag{37}$$

The corresponding generalized Gaudin Hamiltonian in a magnetic field acquires the following explicit form:

$$\hat{H}^l = \sum_{k=1,k\neq l}^{N}\frac{1}{(\nu_k-\nu_l)}\sum_{i,j=1}^{n}\frac{(\nu_k-a_i)}{(\nu_l-a_i)}\hat{S}_{ij}^k\hat{S}_{ji}^l + \frac{1}{2}\sum_{i,j=1}^{n}\frac{1}{(\nu_l-a_i)}\left(\hat{S}_{ij}^l\hat{S}_{ji}^l + \hat{S}_{ji}^l\hat{S}_{ij}^l\right) + \sum_{i,j=1}^{n}c_{ij}(\nu_l)\hat{S}_{ij}^l,$$

where $c(\nu_l) = \sum_{i,j=1}^{n}c_{ij}(\nu_l)X_{ij}$ is the generalized shift element and commutation relation among the $gl(n)$ spin operators $\hat{S}_{ij}^p$, $\hat{S}_{kl}^q$ are standard:

$$\left[\hat{S}_{ij}^p, \hat{S}_{kl}^q\right] = \delta^{pq}\left(\delta_{kj}\hat{S}_{il}^p - \delta_{il}\hat{S}_{kj}^p\right).$$

Let us explicitly write the generalized shift element for the r-matrix $r^A(\lambda,\mu)$:

Proposition 3.3. *Let $r^A(\lambda,\mu)$ be defined by formula (36). Then the $gl(n)$-valued function*

$$c(\lambda) = k\sum_{i=1}^{n}\frac{X_{ii}}{(\lambda-a_i)},$$

where $k\in\mathbb{C}$, is a generalized shift element for $r^A(\lambda,\mu)$.

Proof. The proposition is proved by direct verification. Indeed, let us show that matrix $c(\lambda) = \sum_{i=1}^{n}\frac{X_{ii}}{(a_i-\lambda)}$ satisfies differential equation (12). We have

$$[\Omega_{12}, \partial_l c_2(\lambda)] = -\left[\sum_{i,j=1}^{n}X_{ij}\otimes X_{ji}, \sum_{k=1}^{n}\frac{1\otimes X_{kk}}{(\lambda-a_k)^2}\right] = \sum_{i,k=1}^{n}\frac{X_{ik}\otimes X_{ki}}{(\lambda-a_k)^2} - \sum_{j,k=1}^{n}\frac{X_{kj}\otimes X_{jk}}{(\lambda-a_k)^2},$$

$$\left[\left(r_0^A\right)_{12}, c_1(\lambda)\right] = \left[\sum_{i,j=1}^{n} \frac{X_{ij} \otimes X_{ji}}{(\lambda - a_i)}, \sum_{k=1}^{n} \frac{X_{kk} \otimes 1}{(\lambda - a_k)}\right]$$
$$= \sum_{i,j=1}^{n} \frac{X_{ij} \otimes X_{ji}}{(\lambda - a_i)(\lambda - a_j)} - \sum_{j,k=1}^{n} \frac{X_{kj} \otimes X_{jk}}{(\lambda - a_k)^2},$$
$$\left[\left(r_0^A\right)_{21}, c_2(\lambda)\right] = \left[\sum_{i,j=1}^{n} \frac{X_{ij} \otimes X_{ji}}{(\lambda - a_j)}, \sum_{k=1}^{n} \frac{1 \otimes X_{kk}}{(\lambda - a_k)}\right]$$
$$= \sum_{i,j=1}^{n} \frac{X_{ij} \otimes X_{ji}}{(\lambda - a_i)(\lambda - a_j)} - \sum_{i,k=1}^{n} \frac{X_{ik} \otimes X_{ki}}{(\lambda - a_k)^2}.$$

Hence, we obtain that $[\Omega_{12}, \partial_l c_2(\lambda)] = \left[\left(r_0^A\right)_{12}, c_1(\lambda)\right] - \left[\left(r_0^A\right)_{21}, c_2(\lambda)\right]$, i.e. differential equation (12) is satisfied for the above matrix $c(\lambda)$ and any matrix proportional to it.

Proposition is proved. □

4. Conclusion and discussion

In the present paper, we have constructed integrable generalizations (both classical and quantum) of the Gaudin Hamiltonians in an external magnetic field that correspond to general (non-skew-symmetric) classical r-matrices. The role of the magnetic field is played in our case by the generalized 'shift element' from the generalized 'shift of the argument method'. We have constructed several examples of such 'shift elements' for several classes of classical r-matrices. The important feature of our systems is that the external magnetic field is, generally speaking, non-homogeneous.

The interesting open problem is a description of 'magnetic fields'—all possible shift elements for the different skew-symmetric [7] and non-skew-symmetric [8, 17, 18, 32] classical r-matrices. Another interesting and important, from the physical point of view, problem is a simultaneous diagonalization of the constructed quantum Hamiltonians. The work over these problems is now in progress and some results will soon be published [33].

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